

# ON HYPERBOLIC METRIC AND ASYMPTOTICALLY FINITE INVARIANT DIFFERENTIALS IN HOLOMORPHIC DYNAMICS.

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**ABSTRACT.** Given a rational map  $R$ , we consider the complement of the post-critical set  $S_R$ . In this paper we discuss the existence of invariant Beltrami differentials supported on a  $R$  invariant subset  $A$  of  $S_R$ . Under some geometrical restrictions, either on the hyperbolic geometry of  $A$  or on the asymptotic behavior of infinitesimal geodesics of the Teichmüller space of  $S_R$ , we show the absence of invariant Beltrami differentials supported on  $A$ . In particular, we show that if  $A$  has finite hyperbolic area, then  $A$  can not support invariant Beltrami differentials except in the case where  $R$  is a Lattès map.

## 1. INTRODUCTION

This article is a complementary part to the work done in [1] with its own independent interest. We discuss geometric conditions under which there are no invariant Beltrami differentials supported on the dissipative set of a rational map  $R$ .

In this paper we will always assume that the conservative set of the action of  $R$  belongs to the Julia set.

Now, let us introduce the geometric objects to be treated in this paper.

We denote by  $P(R)$  the closure of the postcritical set of  $R$  and consider the surface  $S_R := \bar{\mathbb{C}} \setminus P(R)$ . The surface  $S_R$  is not always connected, however, on each connected component of  $S_R$  we fix a Poincaré hyperbolic metric and denote by  $\lambda$  the family of all these metrics.

Let  $Q(S_R)$  be the subspace of  $L_1(S_R)$  of holomorphic integrable functions on  $S_R$ .

A rational map  $R$  defines a complex Push-Forward map on  $L_1(\mathbb{C})$ , with respect to the Lebesgue measure  $m$ , which is a contracting endomorphism and is called the complex Ruelle-Perron-Frobenius, for shortness Ruelle operator. The Ruelle operator has the following formula:

$$\begin{aligned} R^*(\phi)(z) &= \sum_{y \in R^{-1}(z)} \frac{\phi(y)}{R'(y)^2} R(\zeta) \\ &= \sum_i \phi(\zeta_i)(z) \zeta'_i(z) \end{aligned}$$

where  $\zeta_i$  is any local complete system of branches of  $R^{-1}$ . The space  $Q(S_R)$  is invariant under the action of the Ruelle operator. The Beltrami operator  $Bel : L_\infty(\mathbb{C}) \rightarrow L_\infty(\mathbb{C})$  given by

$$Bel(\mu) = \mu(R) \frac{\overline{R'}}{R'}$$

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is dual to the Ruelle operator acting on  $L_1(\mathbb{C})$ .

The fixed point space  $\text{Fix}(B)$  of the Beltrami operator is called the *space of invariant Beltrami differentials*. An element  $\alpha \in L_\infty(\mathbb{C})$  is called non trivial if and only if the functional given by

$$v_\alpha(\phi) = \int \phi \alpha$$

is non zero on  $Q(S_R)$ . The norm of  $v_\alpha$  in  $Q^*(S_R)$ , for a non trivial element  $\alpha$ , is called the *Teichmüller norm* of  $\alpha$  and it is denoted by  $\|\alpha\|_T$ .

A non trivial element  $\alpha$  is called *extremal* if and only if the  $\|\alpha\|_\infty = \|\alpha\|_T$ .

A sequence of unit vectors  $\{\phi_i\}$  is called a *Hamilton-Krushkal* sequence, for short HK-sequence, for an extremal element  $\alpha$  if and only if

$$\lim_{i \rightarrow \infty} |v_\alpha(\phi_i)| = \|\alpha\|_\infty.$$

A HK sequence  $\{\phi_i\}$  is called *degenerated* if converge to 0 uniformly on compact sets.

Let  $T : \mathcal{B} \rightarrow \mathcal{B}$  be a linear contraction of a Banach space  $\mathcal{B}$ . An element  $b \in \mathcal{B}$  is called *mean ergodic* with respect to  $T$  if and only if the sequence of Cesàro averages with respect to  $T$ , given by  $C_n(b) = \frac{1}{n} \sum_{i=0}^{n-1} T^i(b)$ , forms a weakly precompact family. Indeed (see Krengel [5]), when  $\mathcal{B}$  is weakly complete then, for a mean ergodic element  $b$ , the sequence  $C_n(b)$  converges in norm to its limit, this limit always is a fixed element of  $T$ . If every element  $b \in \mathcal{B}$  is mean ergodic with respect to  $T$  then the operator  $T$  is called mean-ergodic.

By the Bers Representation Theorem, the space  $Q^*(S_R)$  is linearly quasi-isometrically isomorphic to the *Bergman* space  $B(S_R)$  which is the space of holomorphic functions  $\phi$  on  $S_R$  with the norm  $\|\lambda^{-2}\phi\|_{L_\infty(S_R)}$ .

In the case where  $S_R$  has finitely many components, a classical theorem, see for example [8] and references within, states that  $Q(S_R) \subset B(S_R)$  if and only if the infimum of the length of simple closed geodesics is bounded away from 0.

## 2. MAIN THEOREM

Let  $X$  be an  $R$  invariant measurable set, then the set  $W := \bigcup R^{-n}(X)$  is completely invariant. In the following theorem we will only consider Cesàro averages with respect to the Ruelle operator  $R^*$  in  $L_1(W)$ .

**Theorem 1.** *Let  $X$  be an  $R$  invariant measurable subset such that the restriction map  $r(\phi) = \phi|_X$  from  $Q(S_R)$  to  $L_1(X)$  is weakly precompact. Then every  $\phi \in Q(S_R)$  is mean ergodic with respect to  $R^*$  in  $L_1(W)$ .*

*Proof.* If  $X$  is  $R$  invariant then the Ruelle operator  $R^*$  defines an endomorphism of  $L_1(X)$ . Given  $\phi \in Q(S_R)$ , the family of Cesàro averages  $C_n(\phi)$  restricted on  $X$  forms a weakly precompact subset of  $L_1(X)$ . We claim that  $C_n(\phi)$  converges in norm on  $L_1(X)$ . Indeed, first we show that every weak accumulation point of  $C_n(\phi)$  is a fixed point for the Ruelle operator. Let  $f$  be the weak limit of  $C_{n_i}(\phi)$  for some subsequence  $\{n_i\}$ , then  $R^*(f)$  is the weak limit of  $R^*(C_{n_i}(\phi))$ . By the Fatou Lemma

$$\begin{aligned} \int_X |f - R^*(f)| &\leq \liminf \int_X |C_{n_i}(\phi) - R^*(C_{n_i}(\phi))| \\ &\leq \liminf \|C_{n_i}(\phi) - R^*(C_{n_i}(\phi))\|_{L_1(S_R)} \end{aligned}$$

$$\leq \limsup \|C_{n_i}(I - R^*)(\phi)\|_{L_1(S_R)}.$$

But

$$\|C_{n_i}(I - R^*)(\phi)\|_{L_1(S_R)} \leq \frac{2}{n_i} \|\phi\|_{L_1(S_R)}.$$

Then  $f$  is a non zero fixed point of Ruelle operator. As in [7] we have that  $|f|$  defines a finite absolutely continuous invariant measure. Hence, the support of  $f$  is a non trivial subset of the conservative set of  $R$ . By Lyubich's Ergodicity theorem (see [9] and [6]) and the fact that  $X$  does not intersect the postcritical set we have  $X = W = S_R$ . But, McMullen's Theorem (Theorem 3.9 of [9]) implies that in this case  $R$  is a, so called, *flexible Lattès* map. Furthermore, the space  $Q(S_R)$  is finitely dimensional and hence  $R^*$  is a compact endomorphism of  $Q(S_R)$ , it follows that  $R^*$  is mean ergodic on  $Q(S_R)$ .

Therefore, if  $R$  is not a flexible Lattès map then any weak limit of  $C_n(\phi)$  is 0. Since the weak closure of convex bounded sets is equal to the closure in norm of convex bounded sets, we conclude our claim.

Now let  $W_n = R^{-n}(X)$ , one can inductively prove that  $\phi|_{W_n}$  is mean ergodic on  $L_1(W_n)$ . Indeed, let  $\psi_n = \phi|_{W_n}$ , since  $R^* : L_1(W_n) \rightarrow L_1(W_{n-1}) \subset L(W_n)$  and  $R^*(\psi_n) = R^*(\phi)|_{W_{n-1}}$ , then by arguments above we are done.

Now consider  $\phi|_W - \phi|_{W_n}$ , the  $L_1$  norm of this difference converges to 0 in  $L_1(W)$ , since the Cesàro averages does not expand the  $L_1$  norm we have

$$\|C_k(\phi|_W - \phi|_{W_n})\| \leq \|\phi|_W - \phi|_{W_n}\|.$$

Hence  $C_k(\phi|_W)$  converges to 0 and  $\phi$  is mean ergodic on  $L_1(W)$ . □

Now we state our Main Theorem.

**Theorem 2.** *Let  $R$  be a rational map and let  $X \subset S_R$  be an invariant measurable set of positive Lebesgue measure. Assume that the restriction map  $r(\phi) = \phi|_X$  from  $Q(S_R)$  into  $L_1(X)$  is weakly precompact. If  $\mu$  is a non trivial invariant Beltrami differential, then  $m(\text{supp}(\mu) \cap X) > 0$  if and only if  $R$  is a flexible Lattès map.*

*Proof.* Assume that  $R$  is a flexible Lattès map. Then  $R$  is ergodic on the Riemann sphere and therefore the support of any invariant Beltrami differential  $\mu$  is the whole Riemann sphere. Hence, if  $X$  is invariant of positive Lebesgue measure then  $m(\text{supp}(\mu) \cap X) = m(X) > 0$ .

Again let  $W = \bigcup R^{-n}(X)$ . Now let  $\mu$  be a non trivial invariant Beltrami differential supported on  $W$ . If  $R$  is not Lattès, then for any  $\phi \in Q(S_R)$  we have

$$\int_{S_R} \phi \mu = \int_{S_R} \mu C_k(\phi) = \int_W \mu C_k(\phi).$$

By Theorem 1, the right hand side converges to 0 as  $k$  converges to  $\infty$ . Hence  $\int \phi \mu = 0$  for every quadratic differential  $\phi$  and the functional  $\phi \mapsto \int \phi \mu$  is 0 on  $Q(S_R)$ . Which contradicts the assumption that  $\mu$  is non trivial. □

In the proofs of the previous theorems, the only ingredient was the precompactness of the Cesàro averages  $C_n(\phi)$ . Hence, it is enough to assume the weak precompactness only of Cesàro averages on elements of  $Q(S_R)$ . By results of the second author in [7], see also a related work on [1], it is enough to consider the

Cesàro averages of rational functions in  $Q(S_R)$  having poles only on the set of critical values.

### 3. COMPACTNESS

We want to discuss conditions under which the restriction map  $\phi \mapsto \phi|_A$  is weakly precompact. Unfortunately, so far we have not found conditions where the restriction is weakly precompact but not compact. Let us start with the following observations and definitions.

**Definition.** A rational map  $R$  satisfies the  $B$ -condition if and only if for any  $\phi \in Q(S_R)$  we have

$$\|\lambda^{-2}(z)\phi(z)\|_{L^\infty(S_R)} \leq C\|\phi(z)\|_{L^1(S_R)},$$

where  $C$  is a constant independent of  $\phi$ .

In other words, if  $R$  satisfies the  $B$ -condition, then  $Q(S_R) \subset B(S_R)$  and the inclusion map  $Q(S_R) \rightarrow B(S_R)$  is continuous. As it was noted on the introduction, this happens when  $S_R$  has finitely many components and the infimum of the length of the simple closed geodesics is bounded away from 0.

**Proposition 3.** If  $R$  satisfies the  $B$ -condition and  $\lambda(X) < \infty$  then the restriction map is compact.

*Proof.* If  $R$  satisfies the  $B$ -condition then

$$\lambda^{-2}|\phi(z)| \leq \sup_{z \in S_R} |\lambda^{-2}(z)\phi(z)| \leq C\|\phi\|_1,$$

hence  $|\phi(z)| \leq C\|\phi\|_1\lambda^2(z)$ , by Lebesgue Theorem the restriction map is compact.  $\square$

Using Theorem 2 and Proposition 3 we have the following.

**Corollary 4.** If  $R$  satisfies the  $B$ -condition and  $X$  is an invariant set of positive Lebesgue measure with  $\text{Area}_\lambda(X) < \infty$ . If  $\mu$  is a non zero invariant Beltrami differential, then  $m(\text{supp}(\mu) \cap X) > 0$  if and only if  $R$  is a flexible Lattès map.

In general, the finiteness of the hyperbolic area of  $X$  does not imply the finiteness of hyperbolic area of  $W$ . Generically, it could be that the hyperbolic area of  $W$  is infinite regardless of the area of  $X$ . On the other hand, by Corollary 4, if  $R$  satisfies the  $B$ -condition and the hyperbolic area  $\text{Area}_\lambda(J(R))$  is bounded then  $R$  satisfies Sullivan's conjecture. However, in this situation, we believe that the following stronger statement holds true:

The  $\text{Area}_\lambda(J(R)) < \infty$  if and only if either  $m(J(R)) = 0$  or  $R$  is postcritically finite.

In fact, we do not know if the  $B$ -condition is sufficient on this statement.

Now we consider the more general condition when the restriction map  $r_X$  is compact. This condition, in some sense, reflects the geometry of the postcritical set.

On the product  $S_R \times S_R \subset \mathbb{C}^2$  there exist a unique function  $K(z, \zeta)$  which is characterized by the following conditions.

- (1)  $K(\zeta, z) = -\overline{K(z, \zeta)}$
- (2) For any  $\zeta_0 \in S_R$ , the function  $\phi_{\zeta_0}(z) = K(z, \zeta_0)$  belongs to the intersection  $Q(S_R) \cap B(S_R)$ .

- (3) If  $z_0, \zeta_0$  belong to different components of  $S_R$ , then  $K(z_0, \zeta_0) = 0$ .
- (4) The operator  $P(f)(z) = \int \lambda^{-2}(\zeta) K(z, \zeta) f(\zeta) d\zeta d\bar{\zeta}$  from  $L_1(S_R)$  to  $Q(S_R)$  is a continuous surjective projection.

In fact, the function  $K(z, \zeta)$  is defined on any planar hyperbolic Riemann surface  $S$ . In particular, when the surface  $S$  is the unit disk  $\mathbb{D}$  the function  $K(z, \zeta)$  has the formula

$$K(z, \zeta) = \frac{3}{2} \pi i K_{\mathbb{D}}(z, \zeta)^2$$

where  $K_{\mathbb{D}}(z, \zeta) = [\pi(1 - z\bar{\zeta})^2]^{-1}$  is the classical Bergman Kernel function on the unit disk. For further details on these facts see for example Chapter 3, §7 of the book of I. Kra [4].

Now we consider the following function

$$\omega(\zeta, z) = \lambda^{-2}(\zeta) K(z, \zeta)$$

and

$$w(z) = \omega(z, z).$$

The following proposition is a consequence of Hölder inequality and appear as Lemma 2 on Ohtake's paper [11].

**Proposition 5.** *If  $X$  has positive measure and*

$$\int_X |w| < \infty$$

*then the restriction  $r_X : \phi \mapsto \phi|_X$  from  $Q(S_R)$  to  $L_1(X)$  is compact.*

*Proof.* We follow arguments of Lemma 2 in [11]. If  $D$  is a component of  $S_R$ , then by Hölder's inequality as in Lemma 2 of [11], we have that

$$|(\phi|_D)(z)| \leq C |(w|_D)(z)| \int_D |\phi|$$

where the constant  $C$  does not depend on  $D$ . Since  $S_R$  is a countable union of components, then

$$|\phi(z)| \leq C |w| \|\phi(z)\|.$$

As  $w$  is integrable on  $X$  then by applying once again the Lebesgue Theorem we complete the proof.  $\square$

As a consequence we have:

**Corollary 6.** *If  $\int_{J(R)} |w| < \infty$  then  $R$  satisfies Sullivan's conjecture.*

*Proof.* Follows from Theorem 2 and Proposition 5.  $\square$

Remarks:

- (1) If  $R$  satisfies the  $B$  condition then by Classical results, see the comments before Proposition 1 in [10], we have that  $w(z) \leq C \lambda^2(z)$  where  $C$  does not depend on  $z$ . Partially, if  $X$  has bounded hyperbolic area then  $w(z)$  is integrable on  $X$ , hence the conditions of Proposition 5 implies Proposition 3. As it is mentioned in [10], the conditions in Proposition 3 are strictly weaker than conditions of Proposition 5.

- (2) Moreover, by other result of Ohtake (Proposition 3 in [11]) we note that in general, the boundedness of the hyperbolic area is not a quasiconformal invariant.

In other words, Proposition 3 and Proposition 5 states that if  $X$  is completely invariant positive measure set and satisfying an integrability condition then  $X$  can not support extremal differentials with Hamilton-Krushkal degenerated sequences.

Hence, Corollary 4 and Corollary 6, in the case when  $X$  is a completely invariant, derive from results in [1]. Together, the corollaries mean that if a map  $R$  has an invariant line field which does not allow a Hamilton-Krushkal degenerated sequences on  $Q(S_R)$ , then  $R$  is a Lattès map if and only if the postcritical set has Lebesgue measure zero.

Let  $Y_n$  be an exhaustion of  $S_R$  by compact subsets such that the Lebesgue measure of  $Y_{n+1} \setminus Y_n$  converge to zero. Let  $P_n$  be the sequence of restrictions  $P_n : L_1(S_R) \rightarrow L_1(S_R)$  given by  $P_n(f) = \chi_n P(f)$  where  $\chi_n$  is the characteristic function on  $Y_n$ . Immediately from the definition we have the following facts:

- (1) For each  $n$ , the map  $P_n$  is a compact operator.
- (2) The limit

$$\lim_{n \rightarrow \infty} \|P_n(f) - P(f)\|_{L_1(S_R)} \rightarrow 0$$

for all  $f$  on  $L_1(S_R)$ .

We have the following Theorem:

**Theorem 7.** *Let  $\mu \neq 0$  be an extremal invariant Beltrami differential, then the following conditions are equivalent:*

- *The map  $R$  is a flexible Lattès map.*
- *There exist an exhaustion of compact sets  $Y_n$  as defined above such that the following inequality is true:*

$$\inf_n \|P_n - P\|_{L_1(\text{supp}(\mu))} < 1.$$

*Proof.* Assume that  $R$  is a flexible Lattès map, then  $Q(S_R)$  is finitely dimensional then the operators  $P_n$  converge to  $P$  by norm. Hence, the infimum  $\inf_n \|P_n - P\|_{L_1(\text{supp}(\mu))} = 0$ .

Now, let us assume that  $\inf_n \|P_n - P\|_{L_1(\text{supp}(\mu))} < 1$ . We show that this condition implies that  $\mu$  does not accept degenerated Hamilton-Krushkal sequences. Indeed, assume that  $\{\phi_n\}$  is a degenerated Hamilton-Krushkal sequence for  $\mu$ . By assumption, there exist  $n_0$  such that

$$\sup_{f \in L_1(\text{supp}(\mu)), \|f\|=1} \int |P_{n_0}(f) - P(f)| = r < 1.$$

Since  $\phi_n$  is degenerated and by the compactness of  $P_{n_0}$  we have that

$$\lim_{j \rightarrow \infty} \|P_{n_0}(\phi_j)\|_{L_1(S_R)} \rightarrow 0.$$

Hence

$$\begin{aligned} \|\mu\|_\infty &= \lim_j \left| \int \mu \phi_j \right| = \lim_j \left| \int_{\text{supp}(\mu)} \mu \phi_j \right| \\ &= \lim_j \left| \int_{\text{supp}(\mu)} \mu (P_{n_0}(\phi_j) - P(\phi_j)) \right| \end{aligned}$$

$$\leq \|\mu\|_\infty \sup_{f \in L_1(\text{supp}(\mu)), \|f\|=1} \int |P_{n_0}(f) - P(f)| = r\|\mu\|_\infty < \|\mu\|_\infty.$$

Which is a contradiction.

Applying the Corollary 1.5 in [2], the extremal differential  $\mu$  does not accept Hamilton-Krushkal degenerated sequences if and only if there exist  $\phi$  in  $Q(S_R)$  and a suitable constant  $K$  such that

$$v_\mu(\gamma) = K \int \frac{|\phi|}{\phi} \gamma.$$

Hence, for any  $\gamma$  in  $Q(S_R)$  we have

$$\int \frac{|\phi|}{\phi} R^*(\gamma) = \int \frac{|\phi|}{\phi} \gamma$$

and

$$1 = \int \frac{|\phi|}{\phi} R^*(\phi).$$

This implies that

$$\frac{|R^*(\phi)|}{R^*(\phi)} = \frac{|\phi|}{\phi}$$

but since  $\phi$  is holomorphic then  $\phi$  is a non zero fixed point on  $Q(S_R)$ . Using arguments of the proof of Theorem 1 we are done.  $\square$

The following Proposition is an illustration of when the conditions of Theorem 7 are fulfilled.

**Proposition 8.** *If  $R$  is a rational map satisfying the B condition. If  $A$  is a measurable subset of  $S_R$  so that*

$$\int_A \int_{S_R} |K(z, \zeta)| dz \wedge d\bar{z} \wedge d\zeta \wedge d\bar{\zeta} < \infty$$

*then for any exhaustion of  $S_R$  by compact sets  $Y_n$  and operators  $P_n$  defined as above we have  $\lim \|P_n - P\|_{L_1(A)} = 0$ .*

*Proof.* Let  $Y_n$  be an exhaustion of compact sets as above. Since  $K(z, \zeta)$  is absolutely integrable on  $A \times S_R$  then

$$|\chi_n K(z, \zeta)| \leq |K(z, \zeta)|$$

and  $\chi_n K(z, \zeta) \rightarrow K(z, \zeta)$  pointwise on  $A \times S_R$ . By the Lebesgue theorem

$$\inf \int_A \int_{S_R} |K(z, \zeta) - \chi_n K(z, \zeta)| = 0.$$

For all  $\phi \in Q(S_R)$ , we have

$$\begin{aligned} & \|P_n(\phi) - P(\phi)\|_{L_1(A)} \\ & \leq \int_A |P_n(\phi) - P(\phi)| \leq \int_A \int_{S_R} |\lambda^{-2}(\zeta) \phi(\zeta) (K(z, \zeta) - \chi_n K(z, \zeta))| d\zeta dz \\ & \leq \|\lambda^{-2} \phi\|_\infty \int_A \int_{S_R} |K(z, \zeta) - \chi_n K(z, \zeta)| d\zeta dz \end{aligned}$$

which by the  $B$ -condition we have that the latter is

$$\leq C \|\phi\|_{L_1(S_R)} \int_A \int_{S_R} |K(z, \zeta) - \chi_n K(z, \zeta)| d\zeta dz.$$

For some constant  $C$  which does not depend on  $\phi$ .

Now let  $f \in L_1(A)$ , since  $P$  is a projection then  $f = \phi + \omega$  where  $\phi \in Q(S_R)$ ,  $P(\omega) = P_n(\omega) = 0$  and

$$\|\phi\|_{Q(S_R)} \leq \|P\| \|f\|_{L_1(A)}.$$

Hence  $\lim \|P_n - P\|_{L_1(A)} = 0$ .

□

Finally we characterize a Lattès map in terms of the geometry of  $Q^*(S_R)$ . We start with the following definitions.

- Definition.** (1) A set  $L$  in  $Q^*(S_R)$  is called a geodesic ray if  $L$  is an isometric image of the non negative real numbers  $\mathbb{R}_+$ .  
 (2) Let  $L_1$  and  $L_2$  be geodesic rays with parameterizations  $\psi_1 : \mathbb{R}_+ \rightarrow L_1$  and  $\psi_2 : \mathbb{R}_+ \rightarrow L_2$  respectively. The pair of rays  $L_1$  and  $L_2$  are called equivalent if

$$\limsup_{t \rightarrow \infty} \|\psi_1(t) - \psi_2(t)\|_T \leq d < \infty$$

for some  $d$ .

- (3) An element  $v$  in  $Q^*(S_R)$  is called asymptotically finite if the number of equivalence classes of geodesic rays in  $Q^*(S_R)$  containing 0 and  $v$  is finite.

Now we characterize rational maps which have asymptotically finite non trivial invariant Beltrami differentials.

**Theorem 9.** Assume that  $S_R$  is connected and let  $\mu$  be non trivial an invariant Beltrami differential for  $R$  supported on  $S_R$ . Then the functional  $v_\mu(\phi) = \int \phi \mu$  is asymptotically finite if and only if  $R$  is Lattès.

*Proof.* If  $R$  is a Lattès map then  $Q^*(S_R)$  is finitely dimensional and then there is only a unique geodesic ray passing through any pair of points in  $Q^*(S_R)$  see [2] and [3].

Reciprocally, suppose that the functional  $v_\mu$  is asymptotically finite. Let us first assume that  $\|v_\mu\|_{Q^*(S_R)} = \|\mu\|_{L_\infty}$ . Then by Corollary 6.4 in [2], if  $\mu$  accept degenerated Hamilton-Krushkal sequences there exist  $\mathbb{C}$ -linear isometry  $I : \ell_\infty \rightarrow Q^*(S_R)$  such that if  $m$  is the constant sequence with value  $\|\mu\|_\infty$  then  $I(m) = v_\mu$ .

Now let  $\{e_i\}$  be the canonical basis of  $\ell_\infty$ . Then as in [2], we define geodesic rays in  $\ell_\infty$  as follows:

For any  $r \geq \|\mu\|_\infty$  and

$$\psi_{r,i}(t) = \begin{cases} t \cdot m & \text{for } t \leq r. \\ r \cdot m + (t - r)\|\mu\|_\infty e_i & \text{for } t > r. \end{cases}$$

But for all  $i_0, r_1, r_2$ ,

$$\limsup_{t \rightarrow \infty} \|\psi_{i_0, r_1}(t) - \psi_{i_0, r_2}(t)\|_{\ell_\infty} \leq |r_1 - r_2| \|\mu\|_\infty.$$

Also for all  $i \neq j$  and all  $r$  we have

$$\limsup_{t \rightarrow \infty} \|\psi_{j,r}(t) - \psi_{i,r}(t)\|_\infty$$



$$= \limsup_{t \rightarrow \infty} \|\mu\|_{\infty} t \|e_i - e_j\| = \infty.$$

But the existence of the isometry  $I$  gives a contradiction. Hence  $\mu$  does not accept Hamilton-Krushkal degenerated sequences.

Now using similar arguments as in the proof of Theorem 7, we complete the proof in the case where  $\mu$  is extremal.

Finally we show that if  $\mu$  is a non trivial invariant Beltrami differential, then there exist an extremal invariant differential  $\nu$  such that  $v_{\nu}(\gamma) = v_{\mu}(\gamma)$  for all  $\gamma$  in  $Q(S_R)$ .

Indeed, if  $\mu$  is not extremal then by the Banach Extension Theorem and Riesz Representation Theorem there exist another Beltrami differential  $\alpha$  which is extremal satisfying  $\|\alpha\|_{\infty} = \|\mu\|_T < \|\mu\|_{\infty}$  and such that defines the same functional as  $\mu$  in  $Q(S_R)$ . Let  $\beta$  be a  $*$ -weak limit of the Cesàro averages  $C_n(\alpha) = \frac{1}{n} \sum_{i=0}^{n-1} \alpha(R^i) \frac{(\overline{R^i})'}{(\overline{R^i})}$ , then  $\beta(R) \frac{\overline{R}}{R} = \beta$  and  $\|\beta\|_{\infty} \leq \|\alpha\|_{\infty}$ . Then we claim that  $v_{\beta} = v_{\mu}$ . Let  $\{C_{n_i}(\alpha)\}$  be a sequence of averages  $*$ -weakly converging to  $\beta$ . For any  $\gamma \in Q(S_R)$  we have

$$\int \gamma \beta = \lim \int C_{n_i}(\alpha) \gamma$$

by duality the previous limit is equal to

$$\lim \int \alpha \frac{1}{n_i} \sum_{k=0}^{n_i-1} R^{*k}(\gamma) = \lim \int \mu \frac{1}{n_i} \sum_{k=0}^{n_i-1} R^{*k}(\gamma)$$

but  $\mu$  is an invariant differential and again using duality the previous limit becomes

$$\lim \int \mu \gamma = \int \mu \gamma.$$

Hence for any  $\gamma$  in  $Q(S_R)$  we have

$$\int \beta \gamma = \lim \int C_{n_i}(\alpha) \gamma = \int \mu \gamma.$$

Since  $\alpha$  is extremal we have  $\|\beta\|_{\infty} = \|\alpha\|_{\infty} = \|\mu\|_T$ . Thus  $\beta$  is the desired extremal invariant differential.  $\square$

To conclude, let us note that the arguments of the theorems in this paper work for entire and meromorphic functions in the class of Eremenko-Lyubich. This is the class of all entire or meromorphic functions with finitely many critical and singular values. It is not completely clear whether this arguments can be carried on entire or meromorphic functions whose asymptotic value set contains a compact set of positive Lebesgue measure.

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